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Matrix-product ansatz as a tridiagonal algebra

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Abstract
In the matrix-product states approach to interacting multiparticle systems the stationary probability distribution is expressed as a matrix-product state with respect to a quadratic algebra determined by the dynamics of the process. The states involved in the matrix elements are determined by the boundary conditions. This reflects the intriguing feature of open systems that the bulk behaviour in the steady state strongly depends on the boundary rates. Led by the importance of the boundary conditions we consider the boundary operators as generators of a tridiagonal algebra whose irreducible modules are the Askey–Wilson polynomials. The matrices of the matrix-product ansatz obey the tridiagonal algebraic relations as well for particular values of the structure constants. This suggests the formulation of the steady-state properties in terms of noncommutative matrices generating a tridiagonal Askey–Wilson algebra. The previously known representations, both infinite dimensional and finite dimensional ones, are recovered within the tridiagonal framework.

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1. Introduction

Out of the rich variety of phenomena in nature the most interesting occurs in nonequilibrium conditions and their complex behaviour is far from being well understood. Stochastic interacting particle systems [1, 2] received a lot of attention since they provide a way of modelling phenomena such as traffic flow [3], kinetics of biopolymerization [4], interface growth [5]. Among these, the asymmetric simple exclusion process (ASEP) has become a paradigm in nonequilibrium physics due to its simplicity, rich behaviour and wide range of applicability.

A stochastic process is described in terms of a master equation for the probability distribution $P(s_i, t)$ of a stochastic variable $s_i = 0, 1, 2, \ldots, n-1$ at a site $i = 1, 2, \ldots, L$ of a linear chain. A state on the lattice at a time t is determined by the occupation numbers s_i and a transition to another configuration s_i' during an infinitesimal time step $\mathrm{d}t$ is given by the probability $\Gamma(s, s')\,\mathrm{d}t$. The rates $\Gamma \equiv \Gamma_{jl}^{ik}, i, j, k, l = 0, 1, 2, \ldots, n-1$, are assumed to be independent of the position in the bulk. At the boundaries, i.e. sites 1 and L, additional processes can take place with rates L_i^j and R_i^j. Due to probability conservation

$$\Gamma(s, s) = -\sum_{s' \neq s} \Gamma(s', s). \tag{1}$$

The master equation for the time evolution of a stochastic system

$$\frac{\mathrm{d}P(s, t)}{\mathrm{d}t} = \sum_{s'} \Gamma(s, s')P(s', t) \tag{2}$$

can be mapped to a Schroedinger equation in imaginary time for a quantum Hamiltonian with nearest-neighbour interaction in the bulk and single-site boundary terms:

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = -HP(t) \tag{3}$$

where

$$H = \sum_j H_{j,j+1} + H^{(L)} + H^{(R)}. \tag{4}$$

The probability distribution thus becomes a state vector in the configuration space of the quantum spin chain and the ground state of the Hamiltonian, in general non-Hermitian, corresponds to the steady state of the stochastic dynamics where all probabilities are stationary. The mapping provides a connection with integrable quantum spin chains and allows for exact results of the stochastic dynamics with the formalism of quantum mechanics.

A different approach, inspired by the inverse scattering method in the study of integrable systems and developed for the derivation of exact results for the steady-state properties of interacting many particle systems, is the matrix-product ansatz [6, 7]. The idea is that the stationary probability distribution is expressed as a product of (or a trace over) matrices that form a representation of a quadratic algebra $\Gamma_{jl}^{ik} D_i D_k = x_l D_j - x_j D_l;\ i, j, k = 0, 1, \ldots, n-1$. The algebra of the operators D_i is determined by the dynamics of the process while the states involved in the calculation of the matrix elements are determined by the boundary conditions. The algebraic approach provides an economic and convenient technique for the derivation of solvable recursion relations for the steady-state weights and hence for the calculation of the current and the correlation functions. The recursions have been obtained in earlier works [8, 9] for the two-species model, however they were not readily generalized to other models. Besides the formulation for n-species models [7] there was also a generalization of the matrix-product ansatz to the full dynamic stochastic problem [10].

This work is an attempt to better understanding the symmetry properties underlying the algebraic relations of the matrix approach to one-dimensional stochastic exclusion processes. We put an emphasis on the fact that the states involved in the expressions for the matrix elements are determined by the boundary conditions while the operator algebra is determined by the bulk dynamics. This reflects the property of open stochastic systems that, in contrast to equilibrium mechanics, the boundary conditions are of major importance. We consider the boundary operators of the open asymmetric exclusion process as generators of a tridiagonal algebra whose irreducible modules are given in terms of the Askey–Wilson polynomials. The algebra and its representations depend on the boundary parameters. A special case of the boundary algebra is *a tridiagonal algebra generated by the operators* D_i, $i = 0, 1$, which suggests a formulation of the matrix ansatz as a tridiagonal algebra. We show that the previously known representations used in various applications of the matrix-product approach to different cases of the open ASEP are reconstructed within the tridiagonal algebraic formalism.

2. Matrix-product-state approach to diffusion models

For diffusion processes with n species on a chain of L sites with nearest-neighbour interaction with exclusion a site can be either empty or occupied by a particle of a given type. In the set of occupation numbers $(s_1, s_2, \ldots, s_L)$ specifying a configuration of the system $s_i = 0$ if a site i is empty, $s_i = 1$ if there is a first-type particle at a site $i, \ldots, s_i = n-1$ if there is an $(n-1)$ th-type particle at a site i. On successive sites the species i and k exchange places with probability $g_{ik}\mathrm{d}t$, where $i, k = 0, 1, 2, \ldots, n-1$. With $i < k$, g_{ik} are the probability rates of hopping to the left and g_{ki} to the right. The event of exchange happens if out of two adjacent sites one is a vacancy and the other is occupied by a particle or each of the sites is occupied by a particle of a different type. The n-species symmetric simple exclusion process is known as the lattice gas model of particle hopping between nearest-neighbour sites with a constant rate $g_{ik} = g_{ki} = g$. The n-species asymmetric simple exclusion process with hopping in a preferred direction is the driven diffusive lattice gas of particles moving under the action of an external field. The process is totally asymmetric if all jumps occur in one direction only and partially asymmetric if there is a different nonzero probability of both left and right hopping. The number of particles n_i of each species in the bulk is conserved $\sum_{i=0}^{n-1} n_i = L$ and this is the case of periodic boundary conditions. In the case of open systems, the lattice gas is coupled to external reservoirs of particles of fixed density. In most studied examples [7, 11], one considers phase transitions inducing boundary processes [12] when a particle of type k, $k = 1, 2, \ldots, n-1$ is added with a rate L_k^0 and/or removed with a rate L_0^k at the left end of the chain, and it is removed with a rate R_0^k and/or added with a rate R_k^0 at the right end of the chain.

For diffusion processes, the transition rate matrix becomes simply $\Gamma_{ki}^{ik} = g_{ik}$ and the n-species diffusion algebra [13] has the form

$$g_{ik} D_i D_k - g_{ki} D_k D_i = x_k D_i - x_i D_k \tag{5}$$

where g_{ik} and g_{ki} are the positive (or zero) probability rates, $i, k = 0, 1, \ldots, n-1$, and x_i are the representation-dependent parameters. (No summation over repeated indices in equation (5).) The algebra generated by the n elements D_k obeying the $n(n-1)/2$ relations (5) is an associative algebra with a unit e and with a Poincaré–Birkhoff–Witt basis given by the ordered monomials

$$D_{s_1}^{n_1} D_{s_2}^{n_2} \cdots D_{s_l}^{n_l}, \tag{6}$$

where $s_1 < s_2 < \cdots s_l$, $l \geqslant 1$, and $n_1, n_2, \ldots, n_l$ are non-negative integers.

The quadratic algebra has a Fock representation in an auxiliary Hilbert space where the n generators act as operators. For systems with periodic boundary conditions, the stationary probability distribution is related to the expression

$$P(s_1, \ldots, s_L) = \mathrm{Tr}(D_{s_1} D_{s_2} \cdots D_{s_L}). \tag{7}$$

When boundary processes are considered the stationary probability distribution is related to a matrix element in the auxiliary vector space

$$P(s_1, \ldots, s_L) = \langle w | D_{s_1} D_{s_2} \cdots D_{s_L} | v \rangle \tag{8}$$

with respect to the vectors $|v\rangle$ and $\langle w|$, defined by the boundary conditions

$$\langle w | \big(L_i^k D_k + x_i \big) = 0, \qquad \big(R_i^k D_k - x_i \big) | v \rangle = 0 \tag{9}$$

where x sum up to zero, because of the form of the boundary rate matrices

$$L_i^i = -\sum_{j=0}^{L-1} L_j^i, \qquad R_i^i = -\sum_{j=0}^{L-1} R_j^i, \qquad \sum_{i=0}^{n-1} x_i = 0. \tag{10}$$

The vectors $\langle w|$ and $|v\rangle$ belong to the auxiliary Hilbert space and obey the condition $\langle w|v \neq 0$ in order that the steady states (8) be nonzero. The above relations simply mean that one associates with an occupation number s_i at position i a matrix $D_{s_i} = D_k$ ($i = 1, 2, \ldots, L$; $k = 0, 1, \ldots, n-1$) if a site i is occupied by a k-type particle. The number of all possible configurations of an n-species stochastic system on a chain of L sites is n^L and this is the dimension in the configuration space of the stationary probability distribution as a state vector. Each component of this vector, i.e. the (unnormalized) steady-state weight of a given configuration, is a trace or an expectation value in the auxiliary space given by (7) or (8). The quadratic algebra reduces the number of independent components to only monomials symmetrized upon using relations (5). In various applications of the matrix-product ansatz (MPA) mostly infinite-dimensional representations are used. In the case of the totally asymmetric exclusion process there exist no finite-dimensional representations with dimension bigger than 1. Finite-dimensional representations [14, 15] are defined by a relation between the boundary parameters and the bulk parameter. They correspond to an invariant subspace of the infinite-dimensional matrices and give exact results only on some special curves of the phase diagram. Due to the constraint on the model parameters they restrict the physical properties of the nonequilibrium system in consideration. An example is the three species ASEP with shock profiles [16] where for a constraint on the model parameters the representation of the quadratic algebra is finite dimensional (dimension two) and the stationary state becomes a Bernoulli measure. Finite-dimensional representations are appropriate for the ASEP on a ring [11] as they were suggested [17] for the MPA relation to Bethe ansatz.

3. The quadratic algebra of the asymmetric exclusion process as a tridiagonal algebra

We consider now the two-species partially asymmetric simple exclusion process with incoming and outgoing particles at both boundaries. We simplify the notations, namely, at the left boundary a particle can be added with probability $\alpha \mathrm{d}t$ and removed with probability $\gamma \mathrm{d}t$, and at the right boundary it can be removed with probability $\beta \mathrm{d}t$ and added with probability $\delta \mathrm{d}t$. The system is described by the configuration set $s_1, s_2, \ldots, s_L$ where $s_i = 0$ if a site $i = 1, 2, \ldots, L$ is empty and $s_i = 1$ if a site i is occupied by a particle. The particles hop with a probability $g_{01}\mathrm{d}t$ to the left and with a probability $g_{10}\mathrm{d}t$ to the right, where without loss of generality we can choose the right probability rate $g_{10} = 1$ and the left probability rate $g_{01} = q$. The model depends on five parameters—the bulk probability rate q and the four boundary rates. The asymmetric exclusion process has a particle–hole symmetry

$$\alpha \leftrightarrow \gamma, \qquad \beta \leftrightarrow \delta, \qquad q \leftrightarrow q^{-1} \tag{11}$$

and a left–right symmetry

$$\alpha \leftrightarrow \delta, \qquad \beta \leftrightarrow \gamma, \qquad q \leftrightarrow q^{-1}. \tag{12}$$

The totally asymmetric process corresponds to $q = 0$. The quadratic algebra

$$D_1 D_0 - q D_0 D_1 = x_0 D_1 - D_0 x_1, \qquad x_0 + x_1 = 0 \tag{13}$$

is solved [18] by a pair of deformed oscillators [19, 20] $aa^+ - qa^+a = 1$:

$$D_0 = \frac{x_0}{1-q} + \frac{x_0 a^+}{\sqrt{1-q}}, \qquad D_1 = \frac{-x_1}{1-q} + \frac{-x_1 a}{\sqrt{1-q}}. \tag{14}$$

The boundary conditions have the form

$$(\beta D_1 - \delta D_0)|v\rangle = x_0|v\rangle \qquad \langle w|(\alpha D_0 - \gamma D_1) = \langle w|(-x_1). \tag{15}$$

For a given configuration $(s_1, s_2, \ldots, s_L)$ the stationary probability is given by the expectation value

$$P(s) = \frac{\langle w|D_{s_1} D_{s_2} \cdots D_{s_L}|v\rangle}{Z_L}, \tag{16}$$

where $D_{s_i} = D_1$ if a site $i = 1, 2, \ldots, L$ is occupied and $D_{s_i} = D_0$ if a site i is empty and

$$Z_L = \langle w|(D_0 + D_1)^L|v\rangle \tag{17}$$

is the normalization factor to the stationary probability distribution. The advantage of the matrix-product ansatz is that once the representation of the diffusion algebra and the boundary vectors are known, one can also evaluate all the relevant physical quantities such as the mean density at a site i

$$\langle s_i\rangle = \frac{\langle w|(D_0 + D_1)^{i-1} D_1 (D_0 + D_1)^{L-i}|v\rangle}{Z_L}, \tag{18}$$

the two-point correlation function

$$\langle s_i s_j\rangle = \frac{\langle w|(D_0 + D_1)^{i-1} D_1 (D_0 + D_1)^{j-i-1} D_1 (D_0 + D_1)^{L-j}|v\rangle}{Z_L}, \tag{19}$$

and higher correlation functions. The current J through a bond between site i and site $i + 1$,

$$\begin{aligned} J &= \langle s_i(1 - s_{i+1}) - q(1 - s_i)s_{i+1}\rangle \\ &= \frac{\langle w|(D_0 + D_1)^{i-1}(D_1 D_0 - q D_0 D_1)(D_0 + D_1)^{L-i-1}|v\rangle}{Z_L}, \end{aligned} \tag{20}$$

has a very simple form

$$J = \frac{Z_{L-1}}{Z_L}. \tag{21}$$

One can solve the boundary problem for a process with only incoming particles at the left boundary and only outgoing particles at the right one ($\delta = \gamma = 0$ in (15)) by choosing the vector $|v\rangle$ to be an eigenvector of the annihilation operator a for a real value of the parameter v, $|v\rangle = \mathrm{e}_q^{-\frac{1}{2}vw} \mathrm{e}_q^{va^+}|0\rangle$, and the vector $\langle w|$ to be the eigenvector of the creation operator for the real parameter w, $\langle w| = \langle 0|\, \mathrm{e}_q^{wa}\, \mathrm{e}_q^{-\frac{1}{2}wv}$. According to the algebraic solution, these deformed coherent states [21] are also eigenvectors of the shifted operators D_0, D_1 with the corresponding relations of the eigenvalues

$$\frac{1}{\alpha} = \frac{1}{1 - q} + \frac{w}{\sqrt{1 - q}}, \qquad \frac{1}{\beta} = \frac{1}{1 - q} + \frac{v}{\sqrt{1 - q}}. \tag{22}$$

Due to the proper $q = 0$ limit in this representation the deformed coherent state solution in the case of incoming particles at the left boundary and outgoing particles at the right one provides a unified description [22] of both the partially and the totally asymmetric processes. Alternatively, within the matrix-product approach an exact solution of the ASEP was achieved through relation to q-Hermite polynomials [18] and in a more general case to Al-Chihara polynomials [23]. In the case of general boundary conditions (15) an exact solution related to Askey–Wilson polynomials was proposed and studied [24].

We consider now the algebra generated by three generators D_0, D_1 and their q-commutator $D_2 = [D_0, D_1]_q$, where for any $X, Y [X, Y]_q = q^{1/2}XY - q^{-1/2}YX$.

Proposition 1. *The operators D_0, D_1 and their q-commutator $[D_0, D_1]_q$ form a closed linear algebra*

$$
\begin{aligned}
&[D_0, D_1]_q = D_2 \\
&[D_1, [D_0, D_1]_q]_q = q^{-1/2}x_1(q^{1/2} - q^{-1/2})\{D_0, D_1\} \\
&\qquad - q^{-1}x_1^2 D_0 + q^{-1}x_0x_1D_1 - x_0q^{-1/2}(q^{1/2} - q^{-1/2})D_1^2 \\
&[[D_0, D_1]_q, D_0]_q = -x_0q^{-1/2}(q^{1/2} - q^{-1/2})\{D_0, D_1\} \\
&\qquad - x_0^2q^{-1}D_1 + x_0x_1q^{-1}D_0 - x_1q^{-1/2}(q^{1/2} - q^{-1/2})D_0^2.
\end{aligned}
\tag{23}
$$

The proposition is readily verified by using the explicit form of the MPA quadratic relation (13). The algebra can equivalently be described as a two-relation algebra for the pair D_0, D_1:

$$
\begin{aligned}
&D_0D_1^2 - (q + q^{-1})D_1D_0D_1 + D_1^2D_0 + x_1q^{-1/2}(q^{1/2} - q^{-1/2})\{D_0, D_1\} \\
&\qquad = x_1^2q^{-1}D_0 - x_0x_1q^{-1}D_1 + x_0q^{-1/2}(q^{1/2} - q^{-1/2})D_1^2 \\
&D_0^2D_1 - (q + q^{-1})D_0D_1D_0 + D_1D_0^2 - x_0q^{-1/2}(q^{1/2} - q^{-1/2})\{D_0, D_1\} \\
&\qquad = x_0^2q^{-1}D_1 - x_0x_1q^{-1}D_0 + x_1q^{-1/2}(q^{1/2} - q^{-1/2})D_0^2.
\end{aligned}
\tag{24}
$$

Relations (24) are the well-known Askey–Wilson relations

$$
\begin{aligned}
&A^2A^* - (q + q^{-1})AA^*A + A^*A^2 - \gamma(AA^* + A^*A) = \rho A^* + \gamma^*A^2 + \omega A + \eta \\
&A^{*2}A - (q + q^{-1})A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) = \rho^*A + \gamma A^{*2} + \omega A^* + \eta^*.
\end{aligned}
\tag{25}
$$

The algebra (23) was first considered in the works of Zhedanov [25, 26] who showed that the Askey–Wilson polynomials give pairs of infinite-dimensional matrices satisfying the Askey–Wilson (AW) relations. It is recently discussed in a more general framework of a tridiagonal algebra [27, 28], that is an associative algebra with a unit generated by a (tridiagonal) pair of operators A, A^* and defining relations

$$
\begin{aligned}
&[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \rho A^*] = 0 \\
&[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) - \rho^*A] = 0.
\end{aligned}
\tag{26}
$$

In the general case a tridiagonal pair is determined by the sequence of scalars $\beta, \gamma, \gamma^*, \rho, \rho^*$ from a field K. (We keep the conventional notations, used in the literature, for the scalars of a tridiagonal pair; β and γ should not be confused with the ASEP boundary rates.) Tridiagonal pairs have been classified according to the dependence on the scalars [27]. Examples are the q-Serre relations with $\beta = q + q^{-1}$ and $\gamma = \gamma^* = \rho = \rho^* = 0$

$$
\begin{aligned}
&[A, A^2A^* - (q + q^{-1})AA^*A + A^*A^2] = 0 \\
&[A^*, A^{*2}A - (q + q^{-1})A^*AA^* + AA^{*2}] = 0
\end{aligned}
\tag{27}
$$

and the Dolan–Grady relations [29] with $\beta = 2, \gamma = \gamma^* = 0, \rho = k^2, \rho^* = k^{*2}$

$$
\begin{aligned}
&[A, [A, [A, A^*]]] = k^2[A, A^*] \\
&[A^*, [A^*, [A^*, A]]] = k^{*2}[A^*, A].
\end{aligned}
\tag{28}
$$

Tridiagonal pairs are determined up to an affine transformation

$$
A \to tA + c, \qquad A^* \to t^*A^* + c^*
\tag{29}
$$

where t, t^*, c, c^* are some scalars. The affine transformation can be used to bring a tridiagonal pair in a reduced form with $\gamma = \gamma^* = 0$.

As seen from the Askey–Wilson relations (24) of the ASEP matrices D_0 and D_1, they form a tridiagonal pair with

$$\rho = x_1^2 q^{-1}, \qquad \rho^* = x_0^2 q^{-1}, \qquad \omega = -x_0 x_1 q^{-1} \tag{30}$$

$$\gamma = -x_1 q^{-1/2}(q^{1/2} - q^{-1/2}), \qquad \gamma^* = x_0 q^{-1/2}(q^{1/2} - q^{-1/2}) \tag{31}$$

and $\eta = \eta^* = 0$. Besides $\gamma = \gamma^*, \rho = \rho^*$ due to $x_0 + x_1 = 0$. We can now rescale the operators D_0, D_1 to set $\gamma = \gamma^* = 0$. This is achieved with the help of the transformations

$$D_0 \to D_0 + \frac{x_0 q^{-1/2}}{q^{1/2} - q^{-1/2}}, \qquad D_1 \to D_1 - \frac{x_1 q^{-1/2}}{q^{1/2} - q^{-1/2}}. \tag{32}$$

However the shift of the generators amounts to a tridiagonal pair with sequence of scalars $\beta = -(q + q^{-1}), \gamma = \gamma^* = 0, \rho = \rho^* = 0$. Thus, the operators of the ASEP matrix-product ansatz obey the relations of a tridiagonal algebra

$$\begin{aligned} \left[D_1, D_0 D_1^2 - (q + q^{-1}) D_1 D_0 D_1 + D_1^2 D_0\right] &= 0 \\ \left[D_0, D_1 D_0^2 - (q + q^{-1}) D_0 D_1 D_0 + D_0^2 D_1\right] &= 0 \end{aligned} \tag{33}$$

which is a special case of the tridiagonal relations of the ASEP boundary operators.

4. The tridiagonal boundary algebra

We consider now the general case of incoming and outgoing particles at both boundaries. With all the boundary parameters nonzero there are four operators $\beta D_1, -\delta D_0, -\gamma D_1, \alpha D_0$ and one needs an addition rule to form two linear independent boundary operators acting on the dual boundary vectors. To proceed with a solution we first note that the quadratic algebra is invariant with respect to the following transformations:

$$D_0 \leftrightarrow D_1, \qquad q \leftrightarrow q^{-1} \qquad x_1 \leftrightarrow q^{-1} x_0, \qquad x_0 \leftrightarrow q^{-1} x_1. \tag{34}$$

This symmetry together with

$$\alpha \leftrightarrow q^{-1}\delta \qquad \beta \leftrightarrow q^{-1}\gamma \qquad \gamma \leftrightarrow q^{-1}\beta \qquad \delta \leftrightarrow q^{-1}\alpha \tag{35}$$

$$\alpha \leftrightarrow q^{-1}\gamma, \qquad \beta \leftrightarrow q^{-1}\delta, \qquad \gamma \leftrightarrow q^{-1}\alpha, \qquad \delta \leftrightarrow q^{-1}\beta \tag{36}$$

leaves invariant both the quadratic algebra and the boundary conditions and reflects the left–right and the particle–hole symmetry of the physical system. This results in an isomorphic algebra

$$D_0 D_1 - q^{-1} D_1 D_0 = q^{-1} x_0 D_1 - q^{-1} D_0 x_1 \tag{37}$$

which can be solved by an equivalent set [19, 20] of deformed oscillators

$$\tilde{a}\tilde{a^+} - q^{-1}\tilde{a^+}\tilde{a} = 1. \tag{38}$$

With only two boundary parameters α, β one can use either quadratic algebra to obtain an exactly solvable model through the matrix state method. The situation is different in the four boundary parameter case. One can consider two cases:

(A) Two relations of the same form

$$\beta D_1 \alpha D_0 - q\alpha D_0 \beta D_1 = x_1 \beta\alpha D_0 - \alpha\beta D_1 x_0 \tag{39}$$

and

$$\gamma D_1 \delta D_0 - q\delta D_0 \gamma D_1 = x_1 \gamma\delta D_0 - \delta\gamma D_1 x_0. \tag{40}$$

(B) One relation of the form (39) and another one of the equivalent form

$$\delta D_0\gamma D_1 - q^{-1}\gamma D_1\delta D_0 = q^{-1}x_0\delta\gamma D_1 - q^{-1}\gamma\delta D_0 x_1. \tag{41}$$

These are the two independent relations for the boundary operators corresponding to either case. Any other relation will depend on the representation used for the solution of (39), (40) and (41) in order to be consistent with the solution. In both cases, one needs an addition rule to form two linearly independent boundary operators $B^R = \beta D_1 - \delta D_0$, $B^L = -\gamma D_1 + \alpha D_0$. A solution to this problem within the matrix-product ansatz is obtained by using the $U_q(sl(2))$ algebra in the form of a deformed (u, v) algebra. Some special cases are $U_q(su(2))$ $((u, -u), u < 0)$, a particular q-oscillator algebra $cu_q(2)$ $((u, u), u > 0)$ and two isomorphic oscillator algebras $eu_q^{\pm}(2)$ $(uv = 0)$. The (u, v) deformed algebra is convenient to including all the applications to the solution of the MPA quadratic algebra (13) (or (37)). It is generated by three elements with the defining commutation relations

$$[N, A_\pm] = \pm A_\pm \qquad [A_-, A_+] = uq^N + vq^{-N} \tag{42}$$

and a central element

$$Q = A_+A_- + \frac{vq^N - uq^{1-N}}{1-q}. \tag{43}$$

The representations are labelled by the values of the Casimir

$$Q(\kappa) = \frac{vq^\kappa - uq^{1-\kappa}}{1-q} \tag{44}$$

for some fixed parameter κ. Given a basis $|n, \kappa\rangle$ a representation is defined by $N|n, \kappa\rangle = (\kappa + n)|n, \kappa\rangle$, $A_-|n, \kappa\rangle = r_n|n-1, \kappa\rangle$, $A_+|n, \kappa\rangle = r_{n+1}|n+1, \kappa\rangle$, where

$$r_n^2 = \frac{(1-q^n)(vq^\kappa + uq^{1-n-\kappa})}{1-q}. \tag{45}$$

The state $|0, \kappa\rangle$ is the vacuum with $r_0 = 0$. The representation is infinite dimensional if for all n

$$vq^\kappa + uq^{1-n-\kappa} > 0 \tag{46}$$

which is fulfilled for $U_q(sl_2)$ $(\kappa > 0)$, $cu_q(2)$, $eu_q^{\pm}$ (arbitrary real κ) and finite dimensional of dimension $l + 1$ if (46) is fulfilled for $n < l$, and for some $n = l$

$$vq^\kappa + uq^{-l-\kappa} = 0 \tag{47}$$

which is the case of $U_q(su_2)$.

The deformed (u, v) algebra is an associative algebra and any invertible transformation of the generators is admissible. It can be presented in equivalent forms by means of the transformations

$$\hat{A}_+ = q^{N/2}A_+ \qquad \hat{A}_- = A_-q^{N/2} \tag{48}$$

with

$$[\hat{A}_-, \hat{A}_+]_{q^{-1}} = uq^{2N+1/2} + vq^{1/2} \tag{49}$$

or

$$\tilde{A}_+ = q^{-N/2}A_+ \qquad \tilde{A}_- = A_-q^{-N/2} \tag{50}$$

with

$$[\tilde{A}_-, \tilde{A}_+]_q = uq^{-1/2} + vq^{-2N-1/2}. \tag{51}$$

The (u, u) algebra appears to be more convenient for a solution of the algebraic relations in case B. For case A and in order to emphasize the equivalence of the ASEP to the integrable $su_q(2)$ spin $1/2\,XXZ$ chain we will apply the $(u, -u)$ algebra. Relations (39), (40) can be solved by the corresponding recalling and shift of the following pair of isomorphic deformed operator sets

$$a_1 = q^{N/2}A_+ + \frac{q^{1/2}}{\sqrt{1-q}}q^N \qquad a_1^+ = q^{-N/2}A_+ + \frac{q^{-1/2}}{\sqrt{1-q}}q^{-N} \tag{52}$$

and

$$a_2 = A_-q^{N/2} + \frac{1}{\sqrt{1-q}}q^N \qquad a_2^+ = A_-q^{-N/2} + \frac{1}{\sqrt{1-q}}q^{-N}. \tag{53}$$

This representation allows one to write the boundary operators in the form

$$\begin{aligned} \beta D_1 - \delta D_0 &= -\frac{x_1\beta}{\sqrt{1-q}}q^{N/2}A_+ - \frac{x_0\delta}{\sqrt{1-q}}A_-q^{N/2} - \frac{x_1\beta q^{1/2} + x_0\delta}{1-q}q^N - \frac{x_1\beta + x_0\delta}{1-q} \\ \alpha D_0 - \gamma D_1 &= \frac{x_0\alpha}{\sqrt{1-q}}q^{-N/2}A_+ + \frac{x_1\gamma}{\sqrt{1-q}}A_-q^{-N/2} + \frac{x_0\alpha q^{-1/2} + x_1\gamma}{1-q}q^{-N} + \frac{x_0\alpha + x_1\gamma}{1-q}. \end{aligned} \tag{54}$$

We separate the shift parts from the boundary operators. Denoting the corresponding rest operator parts by A and A^* we write the left and right boundary operators in the form

$$\beta D_1 - \delta D_0 = A - \frac{x_1\beta + x_0\delta}{1-q} \qquad \alpha D_0 - \gamma D_1 = A^* + \frac{x_0\alpha + x_1\gamma}{1-q} \tag{55}$$

Proposition 2. *The operators A and A^* defined by*

$$A = \beta D_1 - \delta D_0 + \frac{x_1\beta + x_0\delta}{1-q} \qquad A^* = \alpha D_0 - \gamma D_1 - \frac{x_0\alpha + x_1\gamma}{1-q} \tag{56}$$

and their q-commutator

$$[A, A^*]_q = q^{1/2}AA^* - q^{-1/2}A^*A \tag{57}$$

form a closed linear algebra

$$\begin{aligned} [[A, A^*]_q, A]_q &= -\rho A^* - \omega A - \eta \\ [A^*, [A, A^*]_q]_q &= -\rho^* A - \omega A^* - \eta^* \end{aligned} \tag{58}$$

where the operator-valued structure constants are given by

$$-\rho = x_0x_1\beta\delta q^{-1/2}(q^{1/2} + q^{-1/2})^2, \qquad -\rho^* = x_0x_1\alpha\gamma q^{-3/2}(q^{1/2} + q^{-1/2})^2 \tag{59}$$

$$\begin{aligned} -\omega &= (x_1\beta q^{1/2} + x_0\delta)(x_1\gamma + x_0\alpha) - \left(x_1^2\beta\gamma + x_0^2\alpha\delta\right)(q^{1/2} - q^{-1/2})Q \\ \eta &= q1/2(q^{1/2} + q^{-1/2})\left(x_0x_1\beta\delta(x_1\gamma + x_0\alpha)Q - \frac{(x_1\beta q^{1/2} + x_0\delta)\left(x_1^2\beta\gamma + x_0^2\alpha\delta\right)}{q^{1/2} - q^{-1/2}}\right) \\ \eta^* &= q1/2(q^{1/2} + q^{-1/2})\left(x_0x_1\alpha\gamma(x_1\beta q^{1/2} + x_0\delta)Q + \frac{(x_0\alpha + x_1\gamma)\left(x_0^2\alpha\delta + x_1^2\beta\gamma\right)}{q^{1/2} - q^{-1/2}}\right). \end{aligned} \tag{60}$$

The proposition is straightforward to verify by using the representation for A and A^* on the RHS of formula (54). We have given the explicit expressions for the structure constants for $U_q(su(2))$ which is relevant for case A. It is a characteristic of this algebra that the structure constants are representation dependent. Analogous expressions are obtained by using any other form of the deformed (u, v) algebra. In particular, for the (u, u) algebra in case B the expressions for ρ, ρ^* differ in sign and we skip the long formulae for ω, η, η^*.

As readily seen from the definition (55), the (shifted) boundary operators of the asymmetric exclusion process obeying the Askey–Wilson algebra (58) form a tridiagonal pair with $\beta = q + q^{-1}, \gamma = \gamma^* = 0$ and $\rho, \rho^*, \omega = \omega^*, \eta, \eta^*$ given by (59) and (60). The tridiagonal algebra of the bulk generators D_0, D_1 is a special form of the Askey–Wilson boundary algebra with $\rho, \rho^* = 0$.

5. Representations of the ASEP tridiagonal algebras

The Askey–Wilson algebra is known to possess some important properties which allow to obtain its ladder representations, spectra, overlap functions. We briefly sketch these properties (for details see [25, 26]). Let f_r be an eigenvector of A with eigenvalue λ_r:

$$A f_r = \lambda_r f_r. \tag{61}$$

Then we can construct a new eigenstate

$$f_s = (Ag(A) + A^* h(A) + A_0 k(A)) f_r \tag{62}$$

where A_0 denotes the q-commutator $[A, A^*]_q$ and

$$A f_s = \lambda_s f_s. \tag{63}$$

It follows from the algebra that f_s will also be an eigenvector of A, if for the new eigenvalue the quadratic relation holds:

$$\lambda_r^2 + \lambda_s^2 - (q + q^{-1})\lambda_r \lambda_s - \rho = 0. \tag{64}$$

This yields for each state f_r two neighbouring states ($r' = r - 1$ and $r'' = r + 1$) whose eigenvalues are the roots of the above quadratic equation. In this parameterization the operator A is diagonal and the operator A^* is tridiagonal:

$$A f_r = a_{r+1} f_{r+1} + b_r f_r + c_{r-1} f_{r-1}. \tag{65}$$

The expressions for the spectrum and the matrix coefficient can be obtained explicitly. The quadratic equation is

$$\lambda_{r+1}^2 + \lambda_r^2 - (q + q^{-1})\lambda_r \lambda_{r+1} - \rho = 0 \tag{66}$$

which yields the spectrum

$$\lambda_r = q^{-r} + \frac{\rho q^r}{(q - q^{-1})^2}. \tag{67}$$

Depending on the sign of ρ it is hyperbolic of the form sh or ch and exp if $\rho = 0$. The algebra possesses a duality property. Due to the duality property the dual basis exists in which the operator A^* is diagonal and the operator A is tridiagonal. We have

$$\begin{aligned} A^* f_p^* &= \lambda_p^* f_p^* \\ A f_s^* &= a_{s+1}^* f_{s+1}^* + b_s^* f_s^* + c_{s-1}^* f_{s-1}^* \end{aligned} \tag{68}$$

where λ_p^* satisfies the quadratic equation (66) with $-\rho$ replaced by $-\rho^*$. The overlap function of the two basis $\langle s|r\rangle = \langle f_s^*|f_r\rangle$ can be expressed in terms of the Askey–Wilson polynomials. It is a long procedure to find the matrix elements, the eigenvalues and the eigenvectors of the operators A, A^* in the ladder representation of the tridiagonal algebra. We have obtained the explicit form of the ASEP boundary algebra infinite-dimensional representations. We have considered two different realizations of the deformed (u, v) algebra for the two different sets of algebraic relations satisfied by the operators $\alpha D_0, \beta D_1, \gamma D_1, \delta D_0$, namely the $(u, -u)$ algebra for case A and the (u, u) algebra for case B. The details are presented elsewhere and we here summarize and comment the results. For further convenience we denote

$$\pm\frac{\gamma}{\alpha} = ac \qquad \pm\frac{\delta}{\beta} = bd \tag{69}$$

where the + sign corresponds to the (u, u) algebra and the − sign to $(u, -u)$ algebra. Besides we set $x_0 = -x_1 = s$ where s is a free parameter from the algebraic relation $x_0 + x_1 = 0$. We rescale the generators as follows:

$$A \to (1-q)\frac{q^{-1/2}}{s\beta}A, \qquad A^* \to (1-q)\frac{q^{-1/2}}{s\alpha}A^*. \tag{70}$$

The representations we have found are isomorphic to the basic representation of the Askey–Wilson algebra [27] in the space of symmetric Laurent polynomials $f[y]$ with a basis $(p_0, p_1, \ldots)$, where $p_n = p_n(x; a, b, c, d)$ denotes the nth Askey–Wilson polynomial [30] depending on four parameters a, b, c, d

$$p_n = {}_4\Phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ay, ay^{-1} \\ ab, ac, ad \end{matrix} | q; q\right) \tag{71}$$

with $p_0 = 1, x = y + y^{-1}$ and $0 < q < 1$. Let $\mathcal{A}$ denote the matrix whose matrix elements enter the three-term recurrence relation for the Askey–Wilson polynomials:

$$xp_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \qquad p_{-1} = 0 \tag{72}$$

$$\mathcal{A} = \begin{pmatrix} a_0 & c_1 & & \\ b_0 & a_1 & c_2 & \\ & b_1 & a_2 & \cdot \\ & & \cdot & \cdot \end{pmatrix}. \tag{73}$$

The explicit form of the matrix elements of A reads

$$a_n = a + a^{-1} - b_n - c_n \tag{74}$$

$$b_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})} \tag{75}$$

$$c_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}. \tag{76}$$

The basis is orthogonal with the orthogonality condition for the Askey–Wilson polynomials $P_n = a^{-n}(ab, ac, ad; q)_n p_n$:

$$\int_{-1}^{1} \frac{w(x)}{2\pi\sqrt{1-x^2}} P_m(x; a, b, c, d|q) P_n(x; a, b, c, d|q)\mathrm{d}x = h_n \delta_{mn} \tag{77}$$

where $w(x) = \frac{h(x,1)h(x,-1)h(x,q^{1/2})h(x,-q^{1/2})}{h(x,a)h(x,b)h(x,c)h(x,d)}$ with $h(x, \mu) = \prod_{k=0}^{\infty}[1 - 2\mu x q^k + \mu^2 q^{2k}]$, and

$$h_n = \frac{(abcdq^{n-1}; q)_n (abcdq^{2n}; q)_\infty}{(q^{n+1}, abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n; q)_\infty}. \tag{78}$$

We summarize the results for the representation of the ASEP boundary operators:

Case A. There is a representation π in a space with basis

$$(p_0, p_1, p_2, \ldots)^t \tag{79}$$

with respect to which the right boundary vector $D_1 - \frac{\delta}{\beta} D_0 \equiv D_1 + bd D_0$ is diagonal. The representing matrix is diag$(\lambda_0, \lambda_1, \lambda_2, \ldots)$ with the eigenvalues λ_n given by

$$\lambda_n = \frac{q^{1/2}}{1-q}(bq^{-n} + dq^n + 1 + bd). \tag{80}$$

The left boundary vector $D_0 - \frac{\gamma}{\alpha} D_1 \equiv D_0 + ac D_1$ is tridiagonal and its representing matrix has the form

$$\pi(D_0 + ac D_1) = \frac{q^{1/2}}{1-q}(b\mathcal{A}^t + 1 + ac). \tag{81}$$

The dual representation π^* has a basis

$$(p_0, p_1, p_2, \ldots) \tag{82}$$

with respect to which the left boundary vector $\pi^*(D_0 + ac D_1)$ is diagonal $(\lambda_0^*, \lambda_1^*, \lambda_2^*, \ldots)$ with diagonal elements

$$\lambda_n^* = \frac{q^{1/2}}{1-q}(aq^{-n} + cq^n + 1 + ac). \tag{83}$$

The right boundary vector is represented by a tridiagonal matrix

$$\pi^*(D_1 + bd D_0) = \frac{q^{1/2}}{1-q}(a\mathcal{A} + 1 + bd). \tag{84}$$

Formulae (80), (81) and (83), (84) define the ladder representation (resp. the dual representation) of the tridiagonal boundary pair in a Hilbert space with an inner product. In the above formulae, the Askey–Wilson parameters are assumed to be arbitrary functions of the boundary parameters except for relations (69). The form of these functions is uniquely determined by the eigenvalue equations with the choice for the left and right boundary vectors:

$$\langle w| = h_0^{-1/2}(p_0, 0, 0, \ldots) \qquad |v\rangle = h_0^{-1/2}(p_0, 0, 0, \ldots)^t \tag{85}$$

where h_0 is a normalization from the orthogonality relation. These vectors belong to the two dual representations of the tridiagonal boundary algebra and are the eigenvectors of the corresponding diagonal operators $\pi(D_1 + bd D_0)$ and $\pi^*(D_0 + ac D_1)$. The eigenvalue equations

$$\left(D_1 - \frac{\delta}{\beta} D_0\right)|v\rangle - s|v\rangle = 0 \qquad \langle w|\left(D_0 - \frac{\gamma}{\alpha} D_1\right) - \langle w|s = 0 \tag{86}$$

are solved by the functions

$$a = \kappa_+^*, \qquad b = \kappa_+, \qquad c = \kappa_-^*, \qquad d = \kappa_- \tag{87}$$

where

$$\begin{aligned} \kappa_\pm &= \frac{-(\beta - \delta - (1-q)) \pm \sqrt{(\beta - \delta - (1-q))^2 + 4\beta\delta}}{2\beta} \\ \kappa_\pm^* &= \frac{-(\alpha - \gamma - (1-q)) \pm \sqrt{(\alpha - \gamma - (1-q))^2 + 4\alpha\gamma}}{2\alpha}. \end{aligned} \tag{88}$$

The expressions on the RHS of (88) have been used in previously known MPA applications, however they have always been taken for granted. It is quite remarkable that within the tridiagonal formulation they arise in a natural way, following from the properties of the AW algebra representations.

Case B. There exist a representation π with basis $(p_0, p_1, \ldots)^t$ with respect to which the right boundary vector $D_1 - \frac{\delta}{\beta} D_0 \equiv \pi(D_1 - bd D_0)$ is diagonal with eigenvalues

$$\lambda_n = \frac{q^{1/2}}{1-q}(bq^{-n} + dq^n + 1 - bd) \tag{89}$$

and the left boundary vector $D_0 - \frac{\gamma}{\alpha} D_1 \equiv \pi(D_0 - acD_1)$ is tridiagonal with

$$\pi(D_0 - acD_1) = \frac{q^{1/2}}{1-q}(b\mathcal{A}^t + 1 - ac). \tag{90}$$

The dual representation has a basis $(p_0, p_1, \ldots)$ with respect to which $\pi^*(D_0 - acD_1)$ is diagonal with diagonal elements

$$\lambda_n^* = \frac{q^{1/2}}{1-q}(aq^{-n} + cq^n + 1 - ac) \tag{91}$$

and $\pi^*(D_1 - bdD_0)$ is tridiagonal

$$\pi^*(D_1 - bdD_0) = \frac{q^{1/2}}{1-q}(a\mathcal{A} + 1 - bd). \tag{92}$$

Once again the relation of the Askey–Wilson parameters to the boundary parameters is uniquely determined by the eigenvalue equations with the choice of the boundary vectors as in (85). The explicit expressions are different from case A and are denoted as

$$a = \kappa_+^{\prime *}, \qquad b = \kappa_+^{\prime}, \qquad c = \kappa_-^{\prime *}, \qquad d = \kappa_-^{\prime} \tag{93}$$

where

$$\begin{aligned} \kappa_\pm^{\prime} &= \frac{-(\beta + \delta - (1-q)) \pm \sqrt{(\beta + \delta - (1-q))^2 - 4\beta\delta}}{2\beta} \\ \kappa_\pm^{\prime *} &= \frac{-(\alpha + \gamma - (1-q)) \pm \sqrt{(\alpha + \gamma - (1-q))^2 - 4\alpha\gamma}}{2\alpha}. \end{aligned} \tag{94}$$

6. The isomorphic tridiagonal algebras of the transfer matrix

The next step is to obtain a representation for the matrix $C = D_0 + D_1$ in terms of which the normalization factor is defined. It enters all the expressions for the physical quantities and plays the role of the transfer matrix. In the tridiagonal formulation the importance of this operator is that it forms an isomorphic tridiagonal pair with each of the boundary operators. This is shown by returning to the set of operators (52) and (53) which provide a solution of the quadratic algebra (13). These imply that we can write the matrix $D_0 + D_1$ in two equivalent forms

$$D_0 + D_1 = \frac{x_0}{\sqrt{1-q}} q^{-N/2} A_+ + \frac{-x_1}{\sqrt{1-q}} A_- q^{-N/2} + \frac{x_0 - x_1}{1-q}(q^{-N} + 1) \tag{95}$$

$$D_0 + D_1 = \frac{x_0}{\sqrt{1-q}} q^{N/2} A_+ + \frac{-x_1}{\sqrt{1-q}} A_- q^{N/2} + \frac{x_0 - x_1}{1-q}(q^{N} + 1). \tag{96}$$

Proposition 3. *The right boundary operator* $B^R = D_1 + bdD_0$ *and the operator* $D_0 + D_1$ *from equation (95), namely the pair*

$$D_1 + bdD_0 = A - \frac{1}{1-q}\left(x_1 + x_0\frac{\delta}{\beta}\right) \qquad D_0 + D_1 = A^* + \frac{1}{1-q}(x_0 - x_1) \tag{97}$$

where now A *and* A^* *obey the tridiagonal algebra relations*

$$\begin{aligned} &[A, A^2A^* - (q+q^{-1})AA^*A + A^*A^2 + bdq^{-1}(q^{1/2} - q^{-1/2})^2 A^*] = 0 \\ &[A^*, A^{*2}A - (q+q^{-1})A^*AA^* + AA^{*2} + (q - q^{-1})^2 A] = 0, \end{aligned} \tag{98}$$

form a tridiagonal pair which is isomorphic to the boundary operators' one. Hence, the representations of this tridiagonal pair are readily obtained from the isomorphic representations of the boundary operators in the basis p_n. Namely, with respect to the basis $(p_0, p_1, p_2, \ldots)^t$ the operator $D_0 + D_1$ is tridiagonal

$$(D_0 + D_1)p_n = \frac{1}{1-q}(2+x)p_n, \qquad xp_n = Ap_n, \qquad \pi(A) = \mathcal{A}^t \tag{99}$$

and it is diagonal with respect to the dual basis $(p_0, p_1, p_2, \ldots)$ with spectrum

$$\lambda^* = \frac{1}{1-q}(2 + q^n - q^{-n}). \tag{100}$$

And, of course, the matrices representing the right boundary vector are the same as in (80) and (84).

Proposition 4. *The left boundary operator $B^L = D_0 - \frac{\gamma}{\alpha}D_1$ and the operator $D_0 + D_1$ in equation (96)*

$$D_0 + D_1 = A + \frac{1}{1-q}(x_0 - x_1) \qquad D_0 - \frac{\gamma}{\alpha}D_1 = A^* + \frac{1}{1-q}\left(x_0 + x_1\frac{\gamma}{\alpha}\right) \tag{101}$$

with

$$\begin{aligned} &[A, A^2A^* - (q+q^{-1})AA^*A + A^*A^2 + (q-q^{-1})^2A^*] = 0 \\ &[A^*, A^{*2}A - (q+q^{-1})A^*AA^* + AA^{*2} + acq^{-1}(q-q^{-1})^2A] = 0 \end{aligned} \tag{102}$$

form a tridiagonal pair which is isomorphic to the boundary operators' one. Hence, the representations of this tridiagonal pair are readily obtained from the isomorphic representations of the boundary operators in the basis p_n. Accordingly, the operator $D_0 + D_1$ is diagonal with respect to the basis $(p_0, p_1, p_2, \ldots)^t$ with eigenvalues

$$\lambda = \frac{1}{1-q}(2 + q^n - q^{-n}) \tag{103}$$

and tridiagonal with respect to the dual basis $(p_0, p_1, p_2, \ldots)$

$$(D_0 + D_1)p_n = \frac{1}{1-q}(2+x)p_n, \qquad xp_n = A^*p_n, \qquad \pi^*(A^*) = \mathcal{A}. \tag{104}$$

The diagonal and tridiagonal representations of the left boundary operator are given in (81) and (83).

Analogous considerations apply for the operator $D_0 + \xi D_1$ (ξ is a fugacity) considered in [24].

Thus, each boundary operator forms a tridiagonal pair with the transfer matrix $D_0 + D_1$. The tridiagonal representation of the transfer matrix with respect to the basis of the Askey–Wilson polynomials which corresponds to the boundary vectors' representation of case A with $\kappa_\pm^{(*)}$ from equation (88) has been used in [24] for the exact solution of the open ASEP. It was considered in the form of an eigenvalue equation for the transfer matrix with the AW polynomials as eigenfunctions, however with no connection to the tridiagonal Askey–Wilson algebra.

7. Finite-dimensional representations of the tridiagonal algebra

The condition for the ladder representations in terms of the Askey–Wilson polynomials to be finite dimensional is $b_n = b_{n_0+n_f} = 0$, where n_f is the dimension of the representation and n_0 is some parameter that is characteristic for the representation. It is appropriate to take n_0 to be equal k, the fixed value of the Casimir characterizing the representation series of the $U_q(sl(2))$ algebra. Without loss of generality we take $n_0 = 0$ for simplicity. Then from the explicit form of b_n it follows that the representation is finite dimensional if for some $n = n_f$ any of the factors

$$1 - abq^n, \qquad 1 - acq^n, \qquad 1 - adq^n, \qquad 1 - abcdq^{n-1} \tag{105}$$

in the numerator of b_n is zero. In particular, the first factor in (77), $1 - abq^n$, expressed in terms of the boundary ASEP parameters, gives

$$\kappa_+^*(\alpha, \gamma)\kappa_+(\beta, \delta) = q^{-n_f} \tag{106}$$

and relabelling $n = 0, 1, 2, \ldots, n_f$ to $l = 1, 2, 3, \ldots, l_f$ we have

$$\kappa_+(\alpha, \gamma)\kappa_+(\beta, \delta) = q^{1-l_f}. \tag{107}$$

The latter relation coincides with the defining condition of a finite-dimensional representation of the operators D_0, D_1 found by Mallick and Sandow in the equivalent realization of a diagonal and an upper diagonal matrix, as they were used in the known applications of the matrix-product ansatz [14, 15].

As already pointed out, the ladder representation of the Askey–Wilson tridiagonal ASEP algebra can be obtained on the basis of either infinite-dimensional representations of some of the different forms of the deformed (u, v) algebra or on the basis of the finite-dimensional representations of $U_q(su(2))$. If one uses the (u, u) algebra or the $(u, 0)$ $((0, u))$ algebra whose representations are infinite dimensional then it is equation (105) only that defines the finite-dimensional representations of the tridiagonal algebra. In the case of a finite-dimensional representation of $U_q(su(2))$ besides the condition for a finite-dimensional representation of the Askey–Wilson algebra following from equation (105) there is one more additional constraint given by equation (47), which is defining the dimension of the $U_q(su(2))$ representation.

8. Special cases of the tridiagonal boundary algebra

From the tridiagonal boundary algebra in the case of general boundary conditions (and the two isomorphic ones involving the transfer matrix operator) one can obtain as special cases the algebras with $\rho = \rho^* = 0$. This is the tridiagonal algebra generated by the MPA matrices D_0, D_1, and also the two isomorphic algebras for the tridiagonal pairs $D_1, D_0 + D_1$ and $D_0 + D_1, D_0$. Such algebras correspond to the open ASEP with only injected particles at the left boundary and only removed particles at the right boundary. It is here important to emphasize that a tridiagonal pair admits a realization as diagonal and tridiagonal matrices and *an equivalent realization as upper and lower bidiagonal matrices* (see [28] for details). In this equivalent form the previously known representations of the MPA matrices D_0, D_1 have been used and are now recovered as the special cases of the ASEP boundary algebra.

(1) $\rho = \rho^* = 0$ due to $ac = 0$, $bd = 0$ and $a \neq 0$, $b \neq 0$.

There is a representation (and its dual) with basis the Al-Chihara polynomials $P_n(a, b; x)$

$$P_n(a, b; x) = a^{-n}(ab; q)_n \times {}_3\Phi_2\left(\begin{matrix} q^{-n}, a\exp^{i\theta}, a\exp^{-i\theta} \\ ab, 0 \end{matrix} |q; q\right) \tag{108}$$

with $P_0 = 1$ and $x = \cos\theta$ and the three-term recurrence relation

$$P_{n+1}(a, b, ; x) + (a + b)q^n P_n(a, b; x) + (1 - q)(1 - abq^{n-1}) P_{n-1}(a, b; x) \\ = 2x P_n(a, b; x). \tag{109}$$

Let $\mathcal{A}_1$ denote the matrix

$$\mathcal{A}_1 = \begin{pmatrix} a_0 & c_1 & & \\ c_1 & a_1 & c_2 & \\ & c_2 & a_2 & \cdot \\ & & \cdot & \cdot \end{pmatrix} \tag{110}$$

satisfying $\mathcal{A}p_n(a, b; x) = 2xp_n(a, b; x)$, where $P_n(a, b; x) = (ab; q)_n p_n(a, b; x)$ and $p_n(a, b; x) = p_n$. Then with respect to $(p_0, p_1, p_2, \ldots)$ $\pi(D_1)$ is diagonal $(\lambda_0, \lambda_1, \lambda_2, \ldots)$

$$\lambda_n = \frac{q^{1/2}}{1 - q}(1 + bq^n) \tag{111}$$

and D_0 is tridiagonal

$$\pi(D_0) = \frac{q^{1/2}}{1 - q}(2 + \mathcal{A}_1). \tag{112}$$

In the dual basis D_0 is diagonal with

$$\lambda_n^* = \frac{q^{1/2}}{1 - q}(1 + aq^n) \tag{113}$$

and D_1 is tridiagonal

$$\pi^*(D_1) = \frac{q^{1/2}}{1 - q}(2 + \mathcal{A}_1). \tag{114}$$

For the isomorphic algebras we write only the matrices representing the transfer matrix $D_0 + D_1$. Namely, both the diagonal and the tridiagonal matrices in the representation π and in the dual representation π^* have the same form

$$\lambda_n = \lambda_n^* = \frac{1}{1 - q}(2 + (a + b)q^n) \tag{115}$$

$$\pi(D_0 + D_1) = \pi^*(D_0 + D_1) = \frac{1}{1 - q}(2 + \mathcal{A}_1) \tag{116}$$

with $\mathcal{A}_1$ from (110). As mentioned above there is an equivalent representation of a tridiagonal pair as a lower and an upper bidiagonal matrix. In this form, the operators D_0, D_1 and with $D_0 + D_1$ as a tridiagonal matrix satisfying the eigenvalue equation

$$(D_0 + D_1)p_n = 2(1 + x)p_n \tag{117}$$

were applied in [23] for the exact solution of the ASEP with only incoming particles at the left boundary and outgoing particle at the right boundary.

(2) $\rho = \rho^* = 0$ and $a = b = c = d = 0$.

For $a = b = c = d = 0$ the Askey–Wilson polynomials reduce to the q-Hermite polynomials

$$H_n(x|q) = P_n(x; 0, 0, 0, 0|q) \tag{118}$$

with the three-term recurrence relation

$$P_{n+1}(x) + (1 - q^n) P_{n-1}(x) = 2x P_n(x). \tag{119}$$

Let $\mathcal{A}_2$ denote the corresponding matrix

$$\mathcal{A}_2 = \begin{pmatrix} 0 & c_1 & & \\ c_1 & 0 & c_2 & \\ & c_2 & 0 & \cdot \\ & & \cdot & \cdot \end{pmatrix} \tag{120}$$

where $c_n = \sqrt{[n]}$ with $[n] = \frac{1-q^n}{1-q}$. The diagonal representation of D_1 and the dual diagonal D_0 are proportional to the infinite-dimensional identity matrices with coefficients determined by the boundary eigenvalue equations. The diagonal transfer matrix is proportional to the infinite-dimensional identity matrix as well. The tridiagonal representation of D_0, D_1 and $D_0 + D_1$ has the form

$$\pi(D_0) = \pi^*(D_1) = \frac{1}{1-q}(1 + \mathcal{A}_2) \tag{121}$$

$$\pi(D_0 + D_1) = \pi^*(D_0 + D_1) = \frac{1}{1-q}(2 + \mathcal{A}_2). \tag{122}$$

As already pointed out a tridiagonal pair is determined by the sequence of scalars by means of which one can construct the representations as a diagonal and a tridiagonal algebra or equivalently as a lower bidiagonal and an upper bidiagonal matrix. In most of the applications of the MPA to different models the representation in the latter form has been used. In the case of the q-Hermite polynomials this equivalent representation corresponds to shifted q-deformed oscillators as they were applied in [18] for the solution of the ASEP with only injected particles on the left boundary and only removed particles on the right one.

9. Discussion and conclusion

In the matrix-product-state ansatz to the open asymmetric exclusion process the two linear independent boundary operators form a tridiagonal pair whose irreducible modules are expressed in terms of the Askey–Wilson polynomials. The four boundary parameters are related to the four parameters of the Askey–Wilson polynomials, the relation being uniquely determined in a given representation by the boundary eigenvalue equations. Each boundary operator forms an isomorphic tridiagonal pair with the operator $C = D_0 + D_1$ with equivalent representations described in terms of the Askey–Wilson polynomials as well. For particular values of the structure constants the matrices D_0, D_1 of the matrix-product ansatz also obey the tridiagonal algebraic relations. Within the tridiagonal framework the known representations of the previously considered MPA applications to the ASEP are recovered as special cases. The tridiagonal Askey–Wilson algebra provides remarkable insight into deep algebraic properties of stochastic processes. Its rich representation theory and the known structure of the polynomials in the Askey–Wilson scheme yield a generalization of the matrix-product approach to a larger variety of possible applications.

The usefulness of the tridiagonal algebraic approach manifests in the simplified calculations of the relevant physical quantities for the ASEP. It is best shown in the two isomorphic algebras involving the transfer matrix $C = D_0 + D_1$. This operator has a special role because it enters all the expressions for the steady weights, the correlation functions, etc, as given by equations (16)–(21). In paper [24], for the study of the open ASEP with most general boundary conditions, two infinite-dimensional tridiagonal matrices D_0, D_1, apparently very complicated, were used to solve the MPA quadratic algebra. The transfer matrix, constructed out of them, satisfies the eigenvalue equation for the AW polynomials, and from our point of view coincides with the tridiagonal representation of either the shifted operator A in (101) or the dual operator A^* in (97), as a generator of the tridiagonal Askey–Wilson algebra. The procedure to compute the current, the steady weights, the correlation functions is simplified by using the orthogonality condition of the Askey–Wilson polynomials and the fact that they form a complete set in the space of Laurent polynomials. Analogous argumentation applies to the study of the ASEP with only incoming particles at the left boundary and outgoing ones at the right one whose solution was related to the q-Hermite and Al-Chihara polynomials [18, 23]. As shown in section 8 the representations of the MPA matrices in the form of upper and lower bidiagonal matrices as they were applied within the MPA in these cases can be reconstructed as the special cases of the tridiagonal AW algebra. Thus, the well-developed representation theory of the Askey–Wilson tridiagonal algebra provides a very elegant and convenient framework which allows for the exact solvability of the open ASEP in the stationary state.

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